Models of Set Theory II - Winter 2013

Prof. Dr. Peter Koepke, Dr. Philipp Schlicht — Problem sheet 6

Problem 21 (4 Points)**.** We work in a ground model M. Suppose that P is a partial order, $\kappa > \omega$ is a cardinal, and $\bar{X} = (X, R_\alpha, f_\alpha)_{\alpha<\kappa}$, $\bar{Y} = (Y, S_\alpha, g_\alpha)_{\alpha<\kappa}$ are structures.

(a) Suppose that $FA_\kappa(P)$ holds, $|X| \leq \omega_1$, and 1_P forces that $\bar{X}$ is embeddable into $\bar{Y}$. Then $\bar{X}$ is embeddable into $\bar{Y}$.

(b) Suppose that $BFA_\kappa(B(P)^*)$ holds, $|X|, |Y| \leq \omega_1$, and 1_P forces that $\bar{X}$ is isomorphic to $\bar{Y}$. Then $\bar{X}$ is isomorphic to $\bar{Y}$.

Problem 22 (6 Points)**.** We work in a ground model M. Suppose that P is a partial order and $\kappa > \omega$ is a cardinal.

(a) Suppose that P is separative and for all $n \in \omega$ and all $p_0, ..., p_n \in P$, there is a greatest lower bound $p_0 \wedge ... \wedge p_n$ whenever there is some $p \leq p_0, ..., p_n$. Show that $BFA_\kappa(B(P)^*)$ implies $BFA_\kappa(P)$.

(b) Let $Col(\omega, \kappa^+) := \{p \colon \omega \to \kappa^+ \mid dom(p) < \omega\}$ and $p \leq q :\Longleftrightarrow p \supseteq q$. Show that $BFA_\kappa(Col(\omega, \kappa^+))$ holds.

(c) Show that
$$H^M_{(\kappa^+)^M} \not\prec_{\Sigma_1} H^{M[G]}_{(\kappa^+)^{M[G]}}$$
for every M-generic filter G on $Col(\omega, \kappa^+)$.

Problem 23 (6 Points)**.** Suppose that M is a ground model and G is Cohen generic over M. Let $f \leq^* g :\Longleftrightarrow \exists m\ \forall n \geq m\ f(n) \leq g(n)$ for functions $f, g \colon \omega \to \omega$.

(a) There is some $g \colon \omega \to \omega$ in $M[G]$ such that $g \not\leq^* f$ for all $f \colon \omega \to \omega$ in M.

(b) There is no $g \colon \omega \to \omega$ in $M[G]$ such that $f \leq^* g$ for all $f \colon \omega \to \omega$ in M.

Problem 24 (4 Points)**.** Let $X \subset^* Y :\Longleftrightarrow X \setminus Y$ is finite and $Y \setminus X$ is infinite, for sets $X, Y \subseteq \omega$. A *tower* is a $\subset^*$-decreasing sequence $(X_\alpha)_{\alpha<\lambda}$ of infinite subsets of ω such that there is no infinite set $X \subseteq \omega$ with $X \subset^* X_\alpha$ for all $\alpha < \lambda$. Let $\mathfrak{t}$ denote the least cardinality of a tower. Show that $\mathfrak{t} \leq \mathfrak{b}$.

Please hand in your solutions on Monday, December 02 before the lecture.